1(a)

We want to solve $y^2 \frac{dy}{dx} = x+1$ subject to $y(0)=1$.

First solve the ODE to get

$$y^2 \frac{dy}{dx} = x+1$$

$$y^2 dy = (x+1)dx$$

$$\int y^2 dy = \int (x+1)\,dx$$

$$\boxed{\frac{y^3}{3} = \frac{x^2}{2} + x + C}$$

Now use $\underbrace{y(0)=1}_{\text{means: plug in } x=0,\, y=1}$ to get:

$$\frac{1^3}{3} = \frac{0^2}{2} + 0 + C$$

$$\boxed{\frac{1}{3} = C}$$

So, $\frac{y^3}{3} = \frac{x^2}{2} + x + \frac{1}{3}$.

Now solve for y to get:

$$y^3 = \frac{3}{2}x^2 + 3x + 1$$

$$y = \left(\frac{3}{2}x^2 + 3x + 1\right)^{1/3}$$

This is valid for all x since you can take the cube root of any number.

Answer:

$$y = \left(\frac{3}{2}x^2 + 3x + 1\right)^{1/3}$$

$$I = (-\infty, \infty)$$

means: $-\infty < x < \infty$

1 (b) We want to solve

$$\boxed{1+\frac{dy}{dx}e^{3x}=0}$$

subject to the condition $\boxed{y(0)=-5}$

First we solve the ODE:

We have $1+\frac{dy}{dx}e^{3x}=0$

$$\frac{dy}{dx}e^{3x}=-1$$

$$dy=\frac{-1}{e^{3x}}dx$$

$$dy=-e^{-3x}\,dx$$

$$\int 1\cdot dy=-\int e^{-3x}\,dx$$

$$y=-\left(\frac{1}{-3}e^{-3x}\right)+C$$

$$\boxed{y=\frac{1}{3}e^{-3x}+C}$$

Now use $\underbrace{y(0)=-5}_{\text{means: plug in } x=0,\ y=-5}$ to get:

$$-5 = y(0) = \frac{1}{3}e^{-3(0)} + c$$

$$-5 = \frac{1}{3}e^{0} + c$$

$e^0 = 1$

$$-5 = \frac{1}{3} + c$$

$$-5 - \frac{1}{3} = c$$

$$c = -\frac{16}{3}.$$

So, $\boxed{y = \frac{1}{3}e^{-3x} - \frac{16}{3}}$

This solution is valid for all x.

Answer

$$y = \frac{1}{3}e^{-3x} - \frac{16}{3}$$

$$I = (-\infty, \infty)$$

means: $-\infty < x < \infty$

1(c) Want to solve $\frac{dy}{dx} = \frac{-x}{y}$ subject to $y(4)=3$.

First solve the ODE:

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\boxed{\frac{y^2}{2} = -\frac{x^2}{2} + C}$$

Now plug in $y(4)=3$ to get:

means: plug in $x=4, y=3$

$$\frac{3^2}{2} = \frac{-(4^2)}{2} + C$$

$$\frac{9}{2} = \frac{-16}{2} + C$$

$$\boxed{\frac{25}{2} = C}$$

Thus,

$$\frac{y^2}{2} = \frac{-x^2}{2} + \frac{25}{2}$$

$$y^2 = -x^2 + 25$$

$$y = \pm\sqrt{-x^2+25}$$

Do we pick + or − ?

We need our function to satisfy $y(4)=3$.

To get $3 = \pm\sqrt{-x^2+25}$ we have to pick the plus sign.

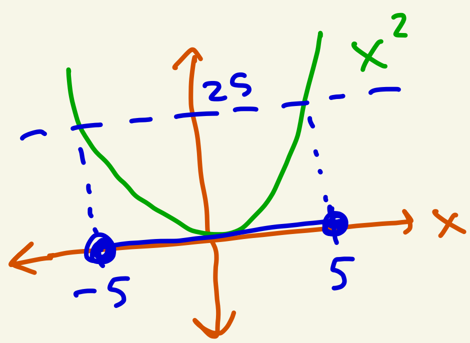
So, $y = \sqrt{-x^2+25}$

Where is this defined?

We need $-x^2+25 \geq 0$.

or $25 \geq x^2$

So, $-5 \leq x \leq 5$.



Answer:

$y = \sqrt{-x^2+25}$

$I = [-5, 5]$

means: $-5 \leq x \leq 5$

① (d) We want to solve $\frac{dy}{dx} = 6y^2x$ subject to $y(0) = \frac{1}{12}$.

First we solve the ODE to get:

$$\frac{dy}{dx} = 6y^2x$$

$$\frac{dy}{y^2} = 6x\,dx$$

$$\int y^{-2}\,dy = \int 6x\,dx$$

$$\frac{y^{-2+1}}{-2+1} = 6\frac{x^2}{2} + C$$

$$\frac{y^{-1}}{-1} = 3x^2 + C$$

$$\boxed{-\frac{1}{y} = 3x^2 + C}$$

Now use $\underbrace{y(0) = \frac{1}{12}}_{\text{means: plug in } x=0,\ y=\frac{1}{12}}$ to get:

$$\left(\frac{-1}{1/12}\right) = 3(0^2) + C$$

$$-12 = C$$

$$\boxed{-12 = c}$$

So, $-\frac{1}{y} = 3x^2 - 12$

Let's solve for y now.

$$\frac{1}{y} = -3x^2 + 12$$

$$y = \frac{1}{-3x^2+12}$$

This function is defined as long as $-3x^2+12 \neq 0$

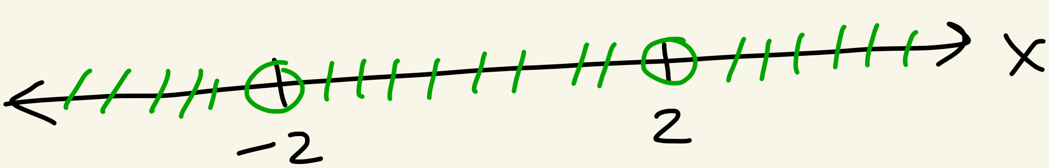
When does $-3x^2+12 = 0$?

When $x^2 - 4 = 0$.

That's when $(x-2)(x+2) = 0$

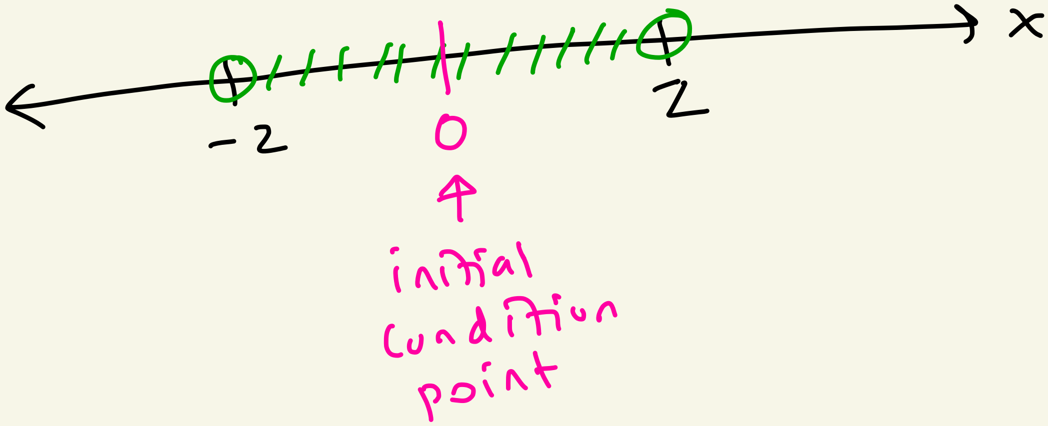
So when $x = 2, -2$.

So, $y = \frac{1}{-3x^2+12}$ is defined when $x \neq 2, -2$.



We want to pick the interval in the

above picture where the initial condition $y(0) = \frac{1}{12}$ lies, that is where $x = 0$ lies.



Answer:

$$y = \frac{1}{-3x^2 + 12}$$

$$I = (-2, 2)$$

Means: $-2 < x < 2$

① (e)

We want to solve $y \frac{dy}{dx} = 3x^2$ subject to $y(0)=2$

First we solve the ODE to get:

$$y \frac{dy}{dx} = 3x^2$$

$$y\,dy = 3x^2\,dx$$

$$\int y\,dy = \int 3x^2\,dx$$

$$\boxed{\frac{y^2}{2} = x^3 + C}$$

Now plug in $\underbrace{y(0)=2}_{\text{means: plug in } x=0,\, y=2}$ to get:

$$\frac{2^2}{2} = 0^3 + C$$

$$\boxed{2 = C}$$

So we get:

$$\frac{y^2}{2} = x^3 + 2$$

$$y^2 = 2x^3 + 4$$

$$y = \pm\sqrt{2x^3+4}$$

First to get $y(0)=2$ we need

$$2 = \pm\sqrt{2(0)^3+4}$$

So we need the + sign.

So, $y = \sqrt{2x^3+4}$

Now for this function to be defined we need $2x^3+4 \geq 0$.

This is when $2x^3 \geq -4$

or when $x^3 \geq -2$

Which is when $x \geq -2^{1/3}$

Answer:

$$y = \sqrt{2x^3+4}$$

$$I = [-2^{1/3}, \infty)$$

$$-2^{1/3} \leq x$$

$\boxed{1(f)}$ We want to solve $xy' = 4y$ subject to $y(1) = 5$.

First we solve the ODE:

$$xy' = 4y$$

$$x \cdot \frac{dy}{dx} = 4y$$

$$\frac{dy}{4y} = \frac{dx}{x}$$

$$\frac{1}{4}\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\boxed{\frac{1}{4}\ln|y| = \ln|x| + C}$$

Now plug in $\underbrace{y(1) = 5}_{}$ to get:

means: plug in $x = 1,\ y = 5$

$$\frac{1}{4}\ln|5| = \ln|1| + C$$

$$\frac{1}{4}\ln(5) = \underbrace{\ln(1)}_{0} + C$$

$$\boxed{C = \frac{1}{4}\ln(5)}$$

So we get:

$$\frac{1}{4}\ln|y| = \ln|x| + \frac{1}{4}\ln(5)$$

$$\ln|y| = 4\ln|x| + \ln(5)$$

$$e^{\ln|y|} = e^{4\ln|x| + \ln(5)}$$

$e^{\ln(A)} = A$

$$|y| = e^{4\ln|x|} \cdot e^{\ln(5)}$$

$A\ln(B) = \ln(B^A)$

$e^{\ln(5)} = 5$

$$|y| = e^{\ln|x|^4} \cdot 5$$

$e^{\ln(A)} = A$

$$|y| = |x|^4 \cdot 5$$

$$|y| = 5|x|^4$$

$$y = \pm 5x^4$$

We need $y(1)=5$, that is we need $5 = \pm 5(1)^4$. So we need to pick the $+$ sign. The solution $y = 5x^4$ is valid for all x.

Answer:

$y = 5x^4$

$I = (-\infty, \infty)$

means: $-\infty < x < \infty$